Modeling Epidemics as Chemical Reaction Processes

Yannis C. Yortsos
and
Assad Oberai
(in collaboration with Harisankar Ramaswamy)

Viterbi vs. Pandemics!

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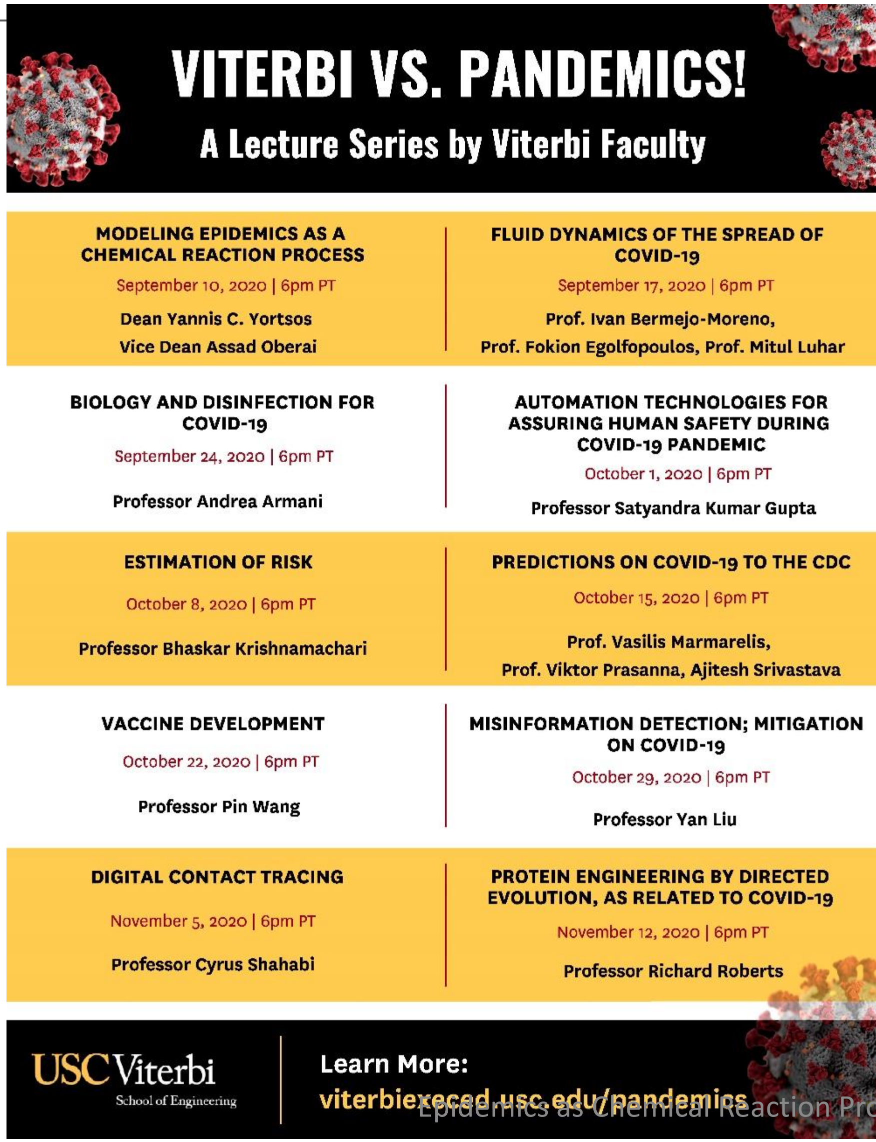
A Series on COVID-19



Lecture Series by Viterbi faculty
TA: Melanie McMullan
macmulla@usc.edu

Human-Human Contagion

-Minimally requires: Susceptible ($\mathcal{S}$), Infected ($\mathcal{I}$), and Recovered ($\mathcal{R}$) (includes perished)*
-Important Variables? Number/Area. Key to infection is *proximity*.
-Need to model how (the rates by which) these populations covert to one another.

$$\mathcal{S} + \mathcal{I} \rightarrow 2\mathcal{I}$$

$$\mathcal{I} \rightarrow \mathcal{R}$$

Analogies with chemical reaction processes

Different sub-populations → chemical species
Number densities (people/area) → molecular concentrations
Infection rates → chemical reaction rates
Spatial transport → advective and diffusive (or dispersive) fluxes

** One can also subdivide further to asymptomatic, secondary infections, etc.*

The General Model

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Advection | Diffusion | Reaction

$$\frac{\partial \mathcal{N}_i}{\partial T} + \boldsymbol{\nabla} \cdot (\boldsymbol{q}\mathcal{N}_i) = -\boldsymbol{\nabla} \cdot (\boldsymbol{\mathcal{D}}_i) + \mathcal{R}_i \qquad (i = S, I, R)$$

$$\boldsymbol{\mathcal{D}}_i = -D\rho\boldsymbol{\nabla}(N_i/\rho)$$

$-\mathcal{N}_i$ is density (number/area) of species i, $\boldsymbol{q}$ is advective velocity

$-\boldsymbol{\mathcal{D}}_i$ is diffusive (or dispersive) flux of i,

$-\mathcal{R}_i$ is reaction rate of species (e.g. that converts populations due to infection)

Also,

$$\mathcal{N}_S + \mathcal{N}_I + \mathcal{N}_R = \rho \quad \text{and} \quad \boldsymbol{\mathcal{D}}_S = \boldsymbol{\mathcal{D}}_I = \boldsymbol{\mathcal{D}}_R = \boldsymbol{\mathcal{D}}$$

Important question: What are the reaction rates? Use *mass-action kinetics*

Infection Rate

$$\mathcal{R}_I = \mathrm{K}\mathcal{N}_S\mathcal{N}_I - \Lambda\mathcal{N}_I\ ;\ \mathcal{R}_S = -\mathrm{K}\mathcal{N}_S\mathcal{N}_I;\ \mathcal{R}_R = -\Lambda\mathcal{N}_I$$

Recovery (or Perished) Rate

"SIR" model, but in terms of areal densities: appropriate for such process

Notes

-Λ is inverse {time} (14/Day): rate, on average, infected individuals recover or die.

-K is inverse {time*(number/area)}: frequency and contact (collisions). K increasing with density (infected and susceptible). Also, contagion is negligible below a certain density (e.g. corresponding to 6 ft). Therefore,

$$\mathrm{K} = \begin{cases} 0; \;\; \rho < \rho_0 \\ \mathrm{K}_0 F\left(\frac{\rho-\rho_0}{\rho_1-\rho_0}\right); \;\; \rho_0 < \rho < \rho_1 \end{cases}$$

where $F(x)$ is a linear function, $F(0) = 0$, $F(1) = 1$; $\rho_0 = 0.1\, m^{-2}$ and $\rho_1 = 1\, m^{-2}$.

-Meaningless to provide area-wide averages (e.g. for states or countries) without differentiating on density (e.g. high density: urban, stadiums, schools, retirement homes; and low density: farms, rural).

-The diffusion coefficient assumes a random walk. For office work, $D = 10^{-3} \frac{m^2}{s}$, two orders of magnitude larger than molecular diffusion in gases.

The Governing Equations

Make things *dimensionless*: Densities normalized by ρ, time by $1/\Lambda$, space by length l, K by K_0, velocities by q. (s, i, r are normalized densities- "probabilities")

$$\frac{\partial s}{\partial t} + (Da\boldsymbol{v} - C\nabla ln\rho)\cdot\nabla s = C\nabla^2 s - R_0(\rho, r)si$$

$$\frac{\partial i}{\partial t} + (Da\boldsymbol{v} - C\nabla ln\rho)\cdot\nabla i = C\nabla^2 i + R_0(\rho, r)si - i$$

$$\frac{\partial r}{\partial t} + (Da\boldsymbol{v} - C\nabla ln\rho)\cdot\nabla r = C\nabla^2 r - i$$

$$\frac{\partial \rho}{\partial t} + Da\boldsymbol{v}\cdot\nabla\rho = 0$$

Defined dimensionless numbers, $Da = \frac{q}{\Lambda l}$ (Damkohler number), $C = \frac{D}{\Lambda l^2} = \varphi^{-2}$ (φ is known as the Thiele modulus) and

$$R_0 = \frac{K_0}{\Lambda}\rho\kappa(\rho, r)$$

$$\kappa(\rho, r) = \begin{cases} 0; & \rho(1-r) < \rho_0 \\ K_0 F\left(\frac{\rho(1-r)-\rho_0}{\rho_1-\rho_0}\right); & \rho_0 < \rho(1-r) < \rho_1 \end{cases}$$

R_0 dependence on density and extent of contagion

The important parameter R_0

1. From

$$\frac{\partial i}{\partial t} + (Da\boldsymbol{v} - C\boldsymbol{\nabla} ln\rho) \cdot \boldsymbol{\nabla} i = C\nabla^2 i + R_0(\rho, r)si - i$$

Initial rate is $(\boldsymbol{R_0}(\boldsymbol{\rho}, \boldsymbol{0}) - \boldsymbol{1})\boldsymbol{i}$

Initial infection grows exponentially, if $\boldsymbol{R_0}(\boldsymbol{\rho}, \boldsymbol{0}) > \boldsymbol{1}$, or decays if $\boldsymbol{R_0}(\boldsymbol{\rho}, \boldsymbol{0}) < \boldsymbol{1}$

2. R_0 depends both on density ρ (number/area) and extent of contagion r

$$\boldsymbol{R_0} = \frac{\boldsymbol{K_0}}{\boldsymbol{\Lambda}} \boldsymbol{\rho\kappa}(\boldsymbol{\rho}, \boldsymbol{r}) \qquad \kappa(\rho, r) = \begin{cases} 0; \; \rho(1-r) < \rho_0 \\ \mathrm{K}_0 F\left(\frac{\rho(1-r)-\rho_0}{\rho_1-\rho_0}\right); \; \rho_0 < \rho(1-r) < \rho_1 \end{cases}$$

3. $R_0(\rho, 0)$ decreases by decreasing ρ (spatial distancing), and/or K_0 (facial covering, isolation of infected, increased air circulation, vaccination), or by increasing Λ (fast recovery)

4. R_0 decreases with extent of contagion and has a final value (at corresponding $R_0(\rho, 0)$)

$$\boldsymbol{R_0}(\boldsymbol{\rho}, \infty) \approx \boldsymbol{R_0}(\boldsymbol{\rho}, \boldsymbol{0})(\boldsymbol{1} - \boldsymbol{r_\infty})$$

Results

A. No entry or exit in or out, constant density, spatially uniform profiles: “Batch reactor” (SIR-like) model

- Infection Curves
- Herd Immunity
- Enforced health policies (e.g. spatial distancing, lockdown)
- Initial conditions; entry in the system for a finite time (“imported infection”)
- “Commuting”

B. Spatially variable interactions: effect of diffusion; infection waves

A. The Batch Reactor (SIR-like) Problem

No spatial gradients; uniform mixing

$$\frac{\partial s}{\partial t} + (Da\boldsymbol{v} - C\boldsymbol{\nabla} ln\rho) \cdot \boldsymbol{\nabla} s = C\nabla^2 s - R_0(\rho, r)si$$

$$\frac{\partial i}{\partial t} + (Da\boldsymbol{v} - C\boldsymbol{\nabla} ln\rho) \cdot \boldsymbol{\nabla} i = C\nabla^2 i + R_0(\rho, r)si - i$$

$$\frac{\partial r}{\partial t} + (Da\boldsymbol{v} - C\boldsymbol{\nabla} ln\rho) \cdot \boldsymbol{\nabla} r = C\nabla^2 r - i$$

$$\frac{\partial \rho}{\partial t} + Da\boldsymbol{v} \cdot \boldsymbol{\nabla}\rho = 0$$

A. The Batch Reactor (SIR-like) Problem (cont.)

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Set of ordinary differential equations

$$s'(t) = -R_0(\rho, r)si$$

$$i'(t) = R_0(\rho, r)si - i$$

$$r'(t) = i$$

$$s + i + r = 1$$

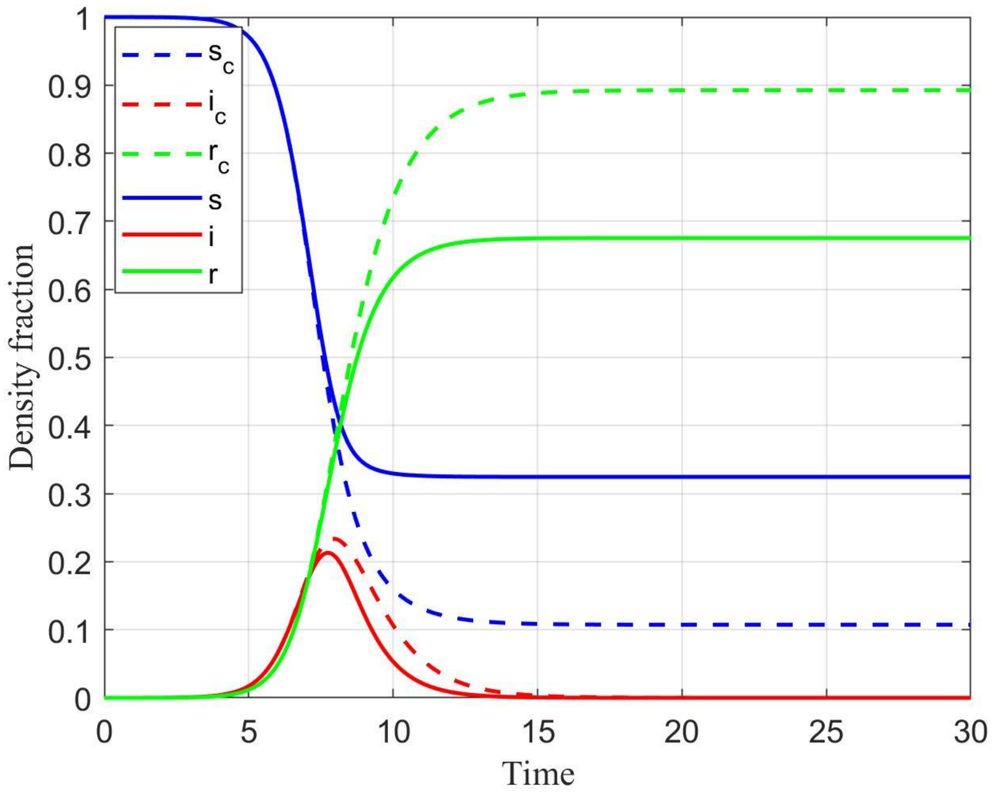
Initial conditions

$$i(0) = i_0; s(0) \equiv s_0 = 1 - i_0;\ r(0) = \ 0$$

Solution requires an initial (even if infinitesimal) seed

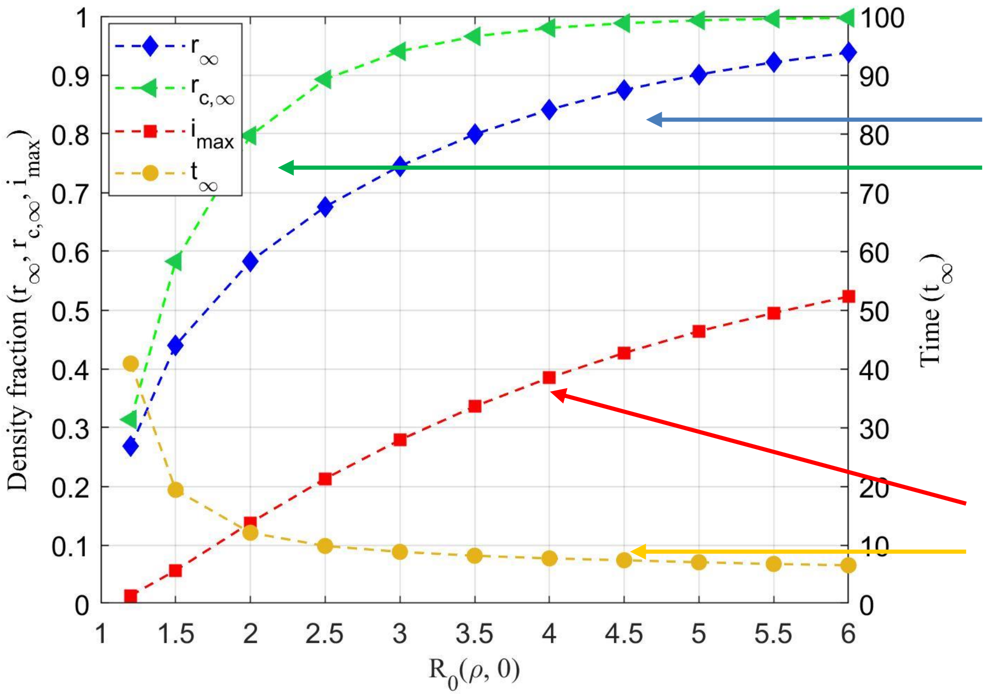
Problem can be solved analytically (closed form expression)

A. The Batch Reactor (SIR-like) Problem: Infection Curves



Infection Curves: $R_0(\rho, r)$ (solid lines); $R_0(\rho, r) = R_0(\rho, 0)$ = 2.5 (dashed lines); $i_0 = 10^{-5}$

A. The Batch Reactor (SIR-like) Problem: Effect of R_0



Herd immunity (variable R_0)

Herd immunity (constant R_0)

Maximum infection fraction

Duration of infection

(t=10 corresponds to 5 months)

1. Herd Immunity is a Function of $\boldsymbol{R_0}$.
2. It always satisfies $R_0(r_\infty)(1 - r_\infty) < 1$, namely it is *stable to perturbations*, but ***not to structural (i.e. R_0) perturbations***. $i'(t) = i\{R_0(\rho, \infty)(1 - r_\infty) - 1\} < 0$
3. Duration of epidemic is longer at lower infection rates.

A. The Batch Reactor (SIR-like) Problem: Effects of Policy

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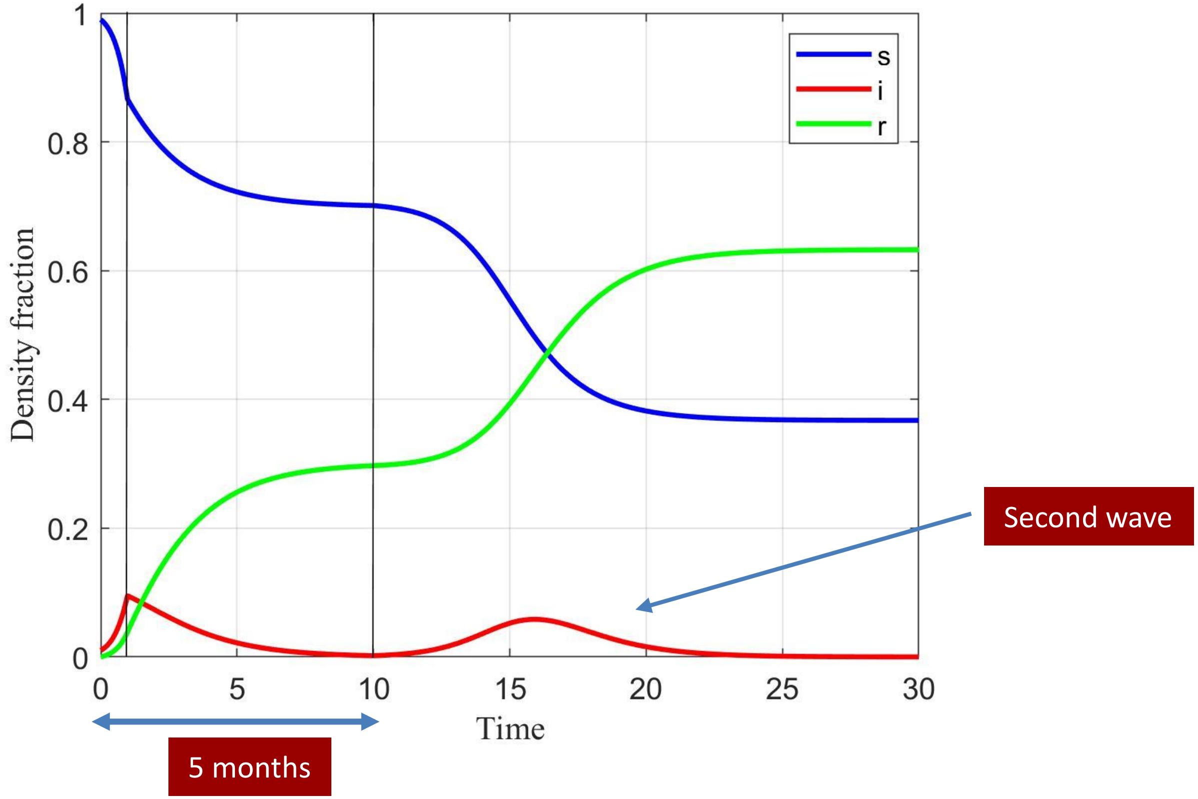
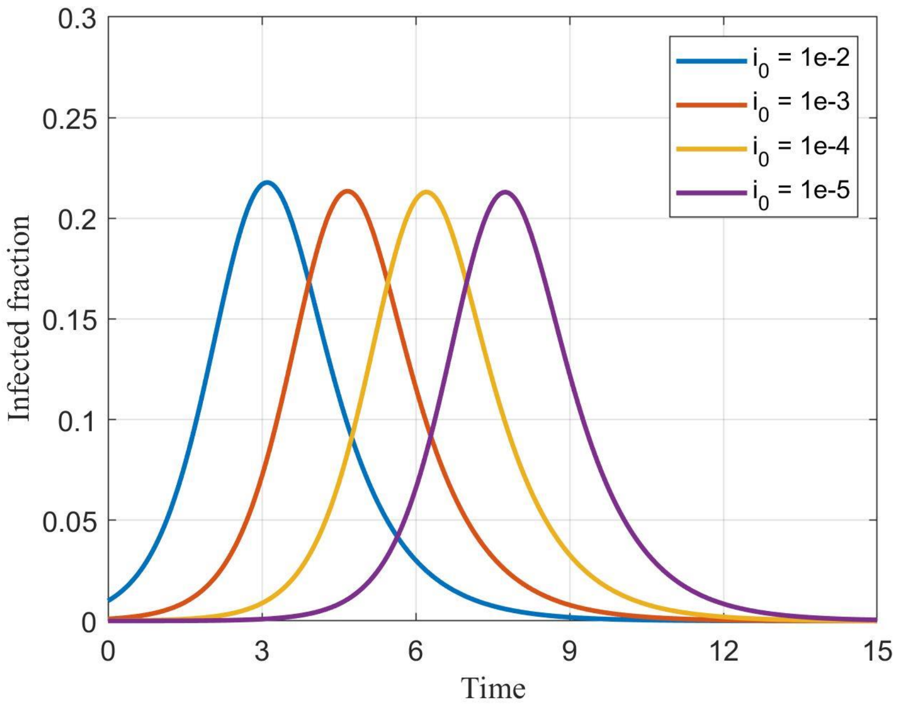
Variation of $\boldsymbol{R_0}$, e.g. through policy (lock-down); effect of relaxing restrictions; and the emergence of “second” wave.



$R_0(\rho, 0)$= 3, t ∈ (0, 1), $R_0(\rho, 0)$ = 0.8, t ∈ (1, 10), $R_0(\rho, 0)$ = 3, t ∈ (10, 30)

A. The Batch Reactor (SIR-like) Problem: Effect of Initial Conditions



1. The effect of initial condition is to simply delay the onset of contagion, all else being equal ($R_0(\rho, 0)$= 2.5). ***Essentially, behavior is solely controlled by R_0.***
2. Similar is the effect of a *travel ban on* imported infections.
3. In either case, contagion is avoided ***only if*** public health policy results into $\boldsymbol{R_0(\rho, 0) < 1}$.

A. The Batch Reactor (SIR-like) Problem: "Commute"

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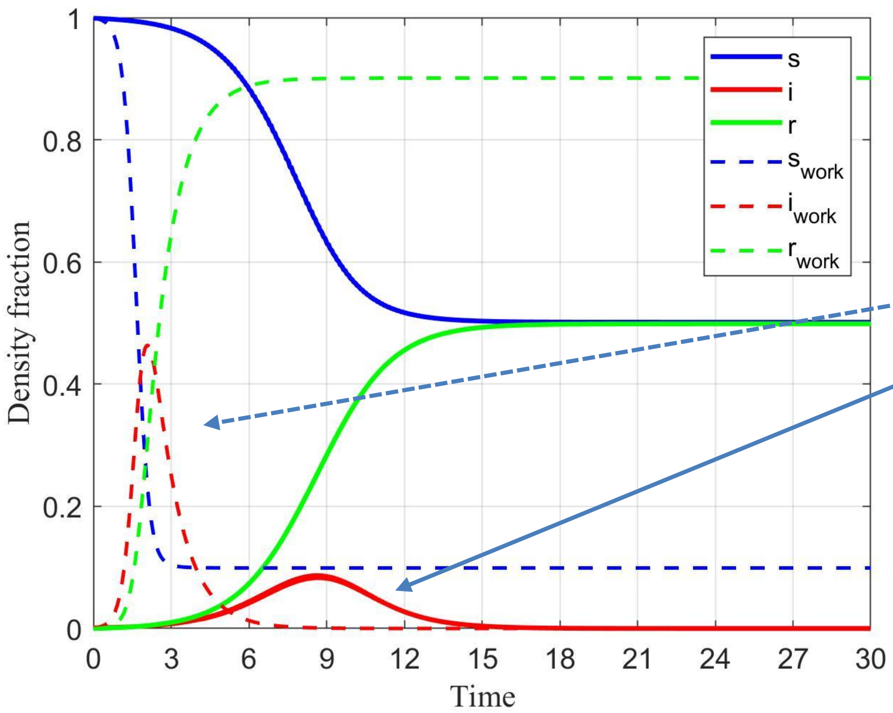
Home
$R_{0,h}(0) = 0$
Time= 1- λ

Work
$R_{0,w}(0) = 5$
Time= λ



$R_{0,w}(\rho,0)$=5
$R_{0,eff}((\rho,0)$=1.66
$\lambda = 1/3$

Commute between "home" and "work" (where $R_{0,h}(\rho,0) = 0$, and $R_{0,w}(\rho,0) > 1$) leads to an effective $\boldsymbol{R_{0,eff} = \lambda R_{0,w}(\rho,0)}$
(equal to the mean value- weighted by the fractional time of exposure λ).

B. Spatially variable interactions: Effects of diffusion

$$\frac{\partial s}{\partial t} + (Da\boldsymbol{v} - C\boldsymbol{\nabla} ln\rho)\cdot \boldsymbol{\nabla} s = C\nabla^2 s - R_0(\rho, r)si$$

$$\frac{\partial i}{\partial t} + (Da\boldsymbol{v} - C\boldsymbol{\nabla} ln\rho)\cdot \boldsymbol{\nabla} i = C\nabla^2 i + R_0(\rho, r)si - i$$

$$\frac{\partial r}{\partial t} + (Da\boldsymbol{v} - C\boldsymbol{\nabla} ln\rho)\cdot \boldsymbol{\nabla} r = C\nabla^2 r - i$$

$$\frac{\partial \rho}{\partial t} + Da\boldsymbol{v}\cdot \boldsymbol{\nabla}\rho = 0$$

Focus on diffusion only

No advection: **Then, ρ is only a function of space (not time)**

Explore effects of diffusion on the onset and propagation of infection waves

B. Spatially variable interactions: Traveling Waves

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Constant ρ, 1-D, steady-states in coordinate $\xi = x - Vt$, where V is wave velocity

$$-V\frac{\partial \bar{s}}{\partial \xi} = C\frac{\partial^2 \bar{s}}{\partial \xi^2} - R_0 \bar{s}\bar{\imath} \qquad -\infty < \xi < \infty$$

$$-V\frac{\partial \bar{\imath}}{\partial \xi} = C\frac{\partial^2 \bar{\imath}}{\partial \xi^2} + R_0 \bar{s}\bar{\imath} - \bar{\imath} \qquad -\infty < \xi < \infty$$

$$-V\frac{\partial \bar{r}}{\partial \xi} = C\frac{\partial^2 \bar{r}}{\partial \xi^2} + \bar{\imath} \qquad -\infty < \xi < \infty$$

No-flux conditions at the ends: $\frac{\partial \bar{s}}{\partial \xi} = \frac{\partial \bar{\imath}}{\partial \xi} = \frac{\partial \bar{r}}{\partial \xi} = 0$ at $\xi = \pm\infty$

Find

$$V = \frac{1}{r_{V,\infty}} \int_{-\infty}^{\infty} \bar{\imath}\, d\xi$$

Questions: 1. Are the profiles the same as for the Batch (SIR) problem?
2. And what is the effect of the diffusion coefficient C?

B. Spatially variable interactions: 1-D Contagion Waves

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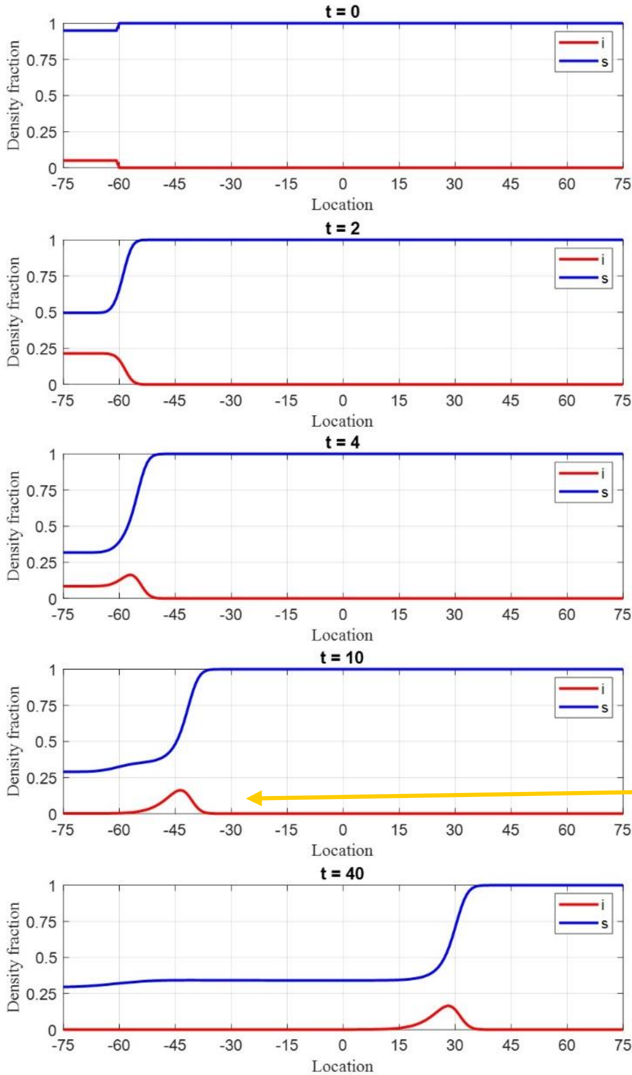
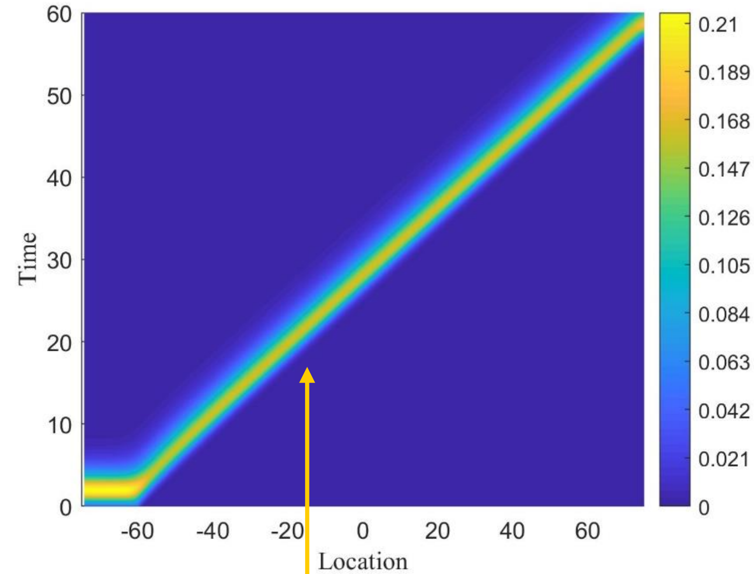
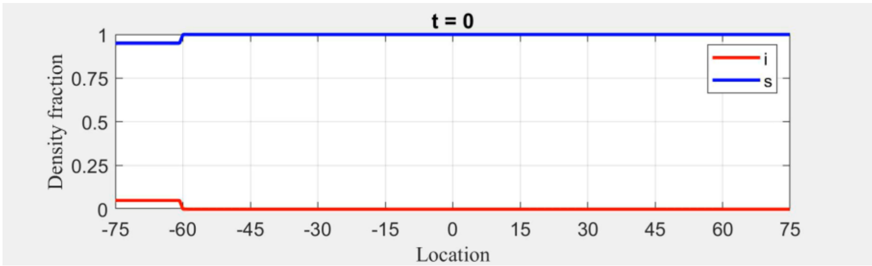


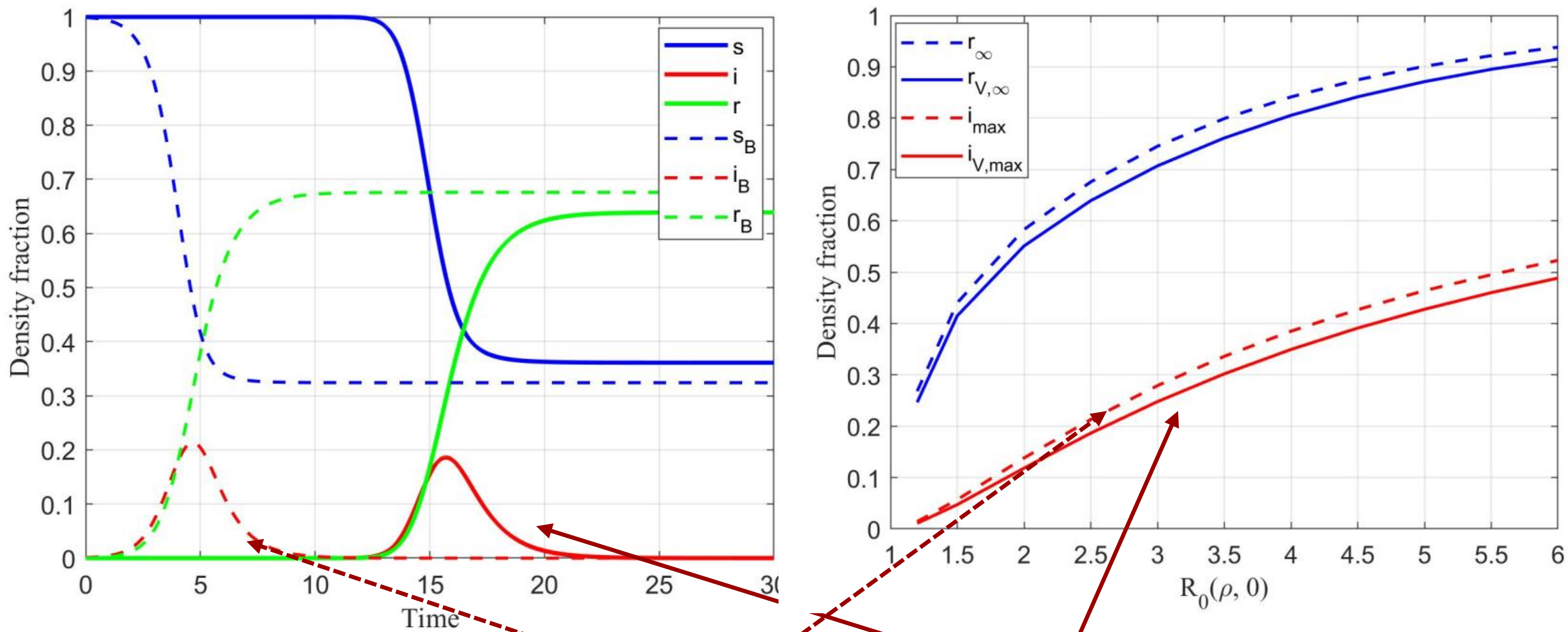
$R_0(\rho, 0)$= 2.5



Infection Wave



B. Spatially variable interactions: Comparison with batch “SIR” model



$R_0(\rho, 0) = 2.5$

Batch (SIR) Problem; Diffusion Included

Effect of diffusion is to slightly lower the equivalent infection rates

B. Spatially variable interactions: Diffusion dependence

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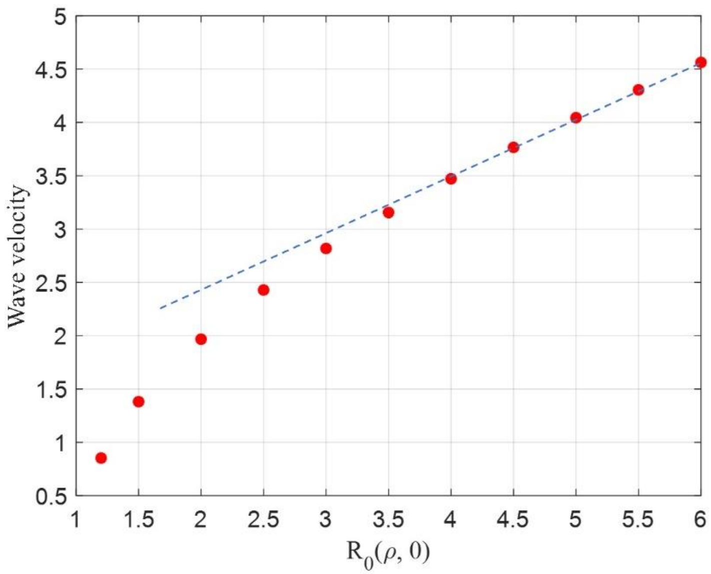
We can explicitly remove the C-dependence by introducing rescaled space coordinates and velocities, $\xi = \sqrt{C}\zeta$ and $V = W\sqrt{C}$.
All equations remain the same, so we can formally take $C = 1$ and derive results independent of C

$$W = \frac{1}{r_{V,\infty}} \int_{-\infty}^{\infty} \bar{i}_1 d\zeta$$

In dimensional form

$$\mathcal{V} = \sqrt{D\Lambda} \frac{1}{r_{V,\infty}} \int_{-\infty}^{\infty} \bar{i}_1 d\zeta$$



Wave velocity increases with the square root of $D\Lambda$ and with $R_0(\rho, 0)$
Same results hold for radial symmetry geometries
Diffusion and reaction lead to translational waves

B. Spatially variable interactions: 1-D Heterogeneity

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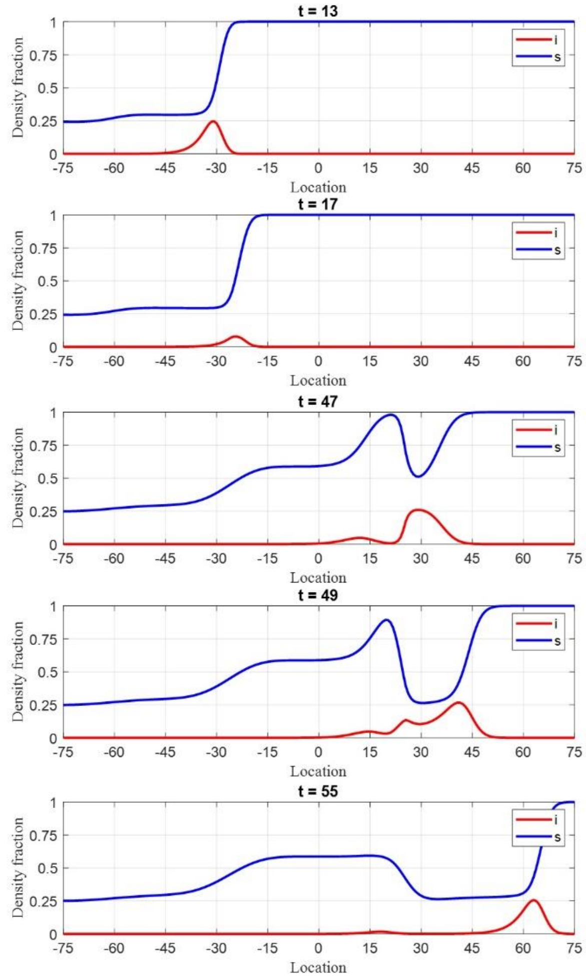
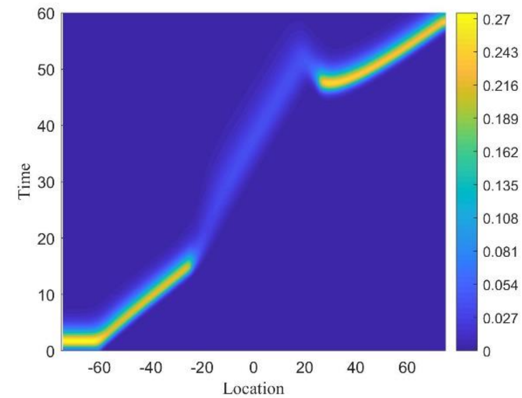
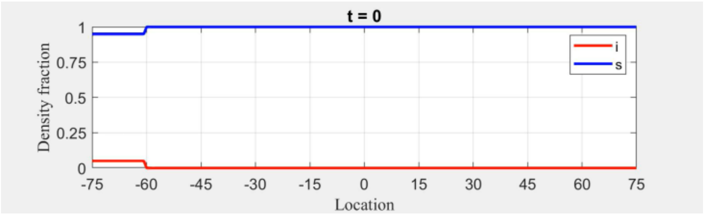


Variable density:

$R_0(\rho, 0)$ = 3, for x ∈ (−80, −25) and x ∈ (25, 80); $R_0(\rho, 0)$ = 1.5 for x ∈ (−25, 25).





Wave velocities and profiles rapidly reach their steady-state values corresponding to the ambient $R_0(\rho, 0)$

B. Spatially variable interactions: 2-D geometries

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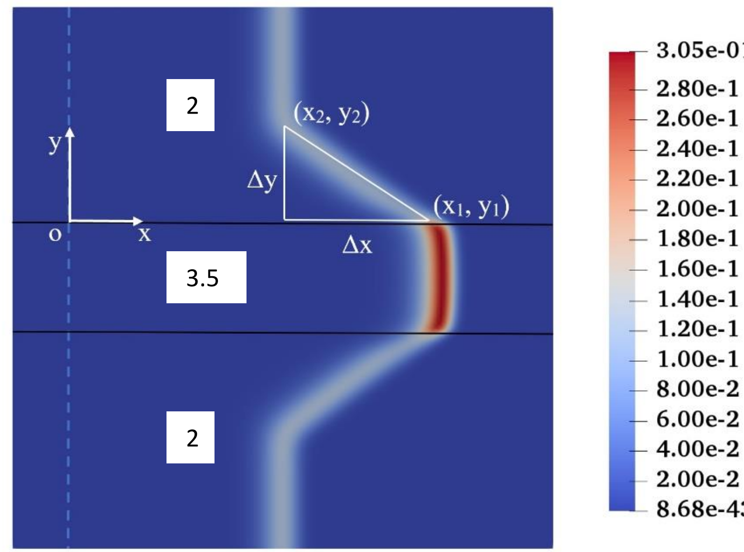
Effect of 2-D heterogeneity in $R_0(\rho, 0)$: 1. Layered System

$R_0(\rho, 0)$ = 2 in the outer layers, and $R_0(\rho, 0)$ = 3.5 in the inner layer.



As in 1-D, wave velocities and profiles rapidly reach their steady-state values corresponding to the ambient $R_0(\rho, 0)$. Connecting wave-fronts are linear.

B. Spatially variable interactions: 2-D geometries

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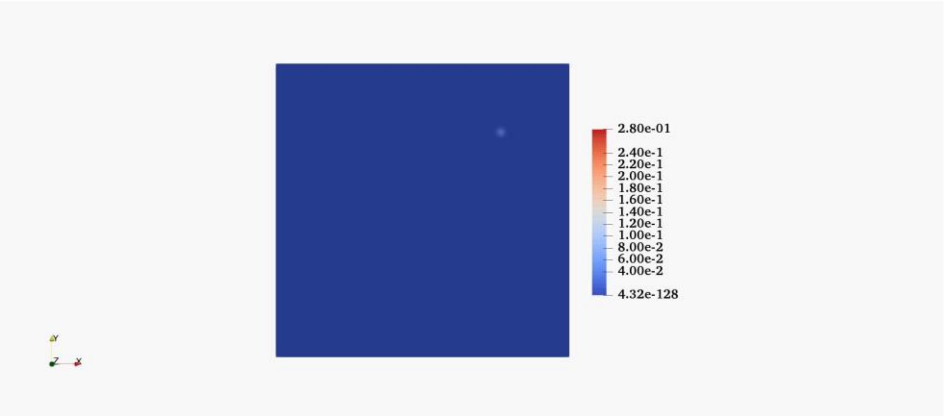
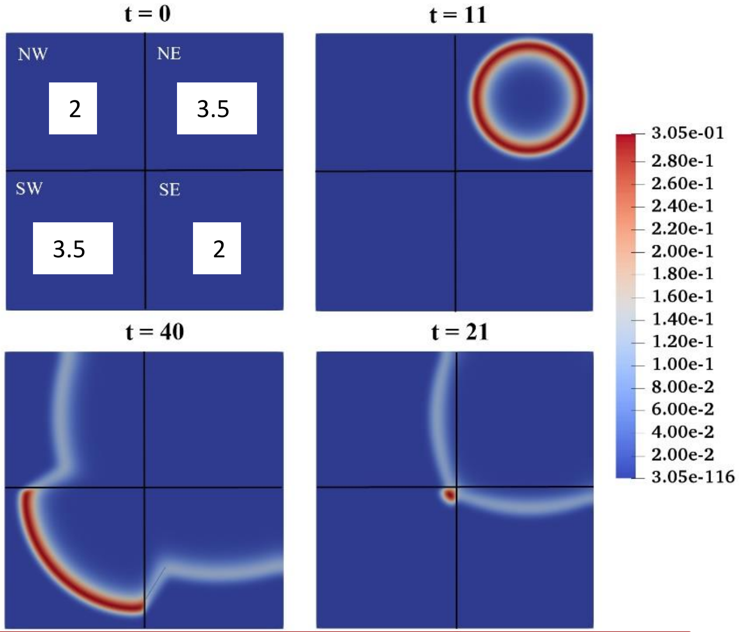
Effect of 2-D heterogeneity in $R_0(\rho, 0)$: 2. 4- Quadrant System

$R_0(\rho, 0)$ = 2 in NW and SE, and $R_0(\rho, 0)$ = 3.5 in NE and SW





As in layered system, wave velocities and profiles rapidly reach their steady-state values corresponding to the ambient $R_0(\rho, 0)$. Connecting wave-fronts are linear.

Concluding Remarks

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- Understanding of the spreading of epidemics can benefit substantially from reaction-diffusion analogies.
- Important to model in terms of spatial densities.
- Kinetics can naturally incorporate spatial distancing.
- Important variable $R_0(\rho, r)$ is a function of spatial density and process extent
- SIR-like model results as the Batch Reactor equivalent.
- Herd immunity is a function of $R_0(\rho, 0)$. It is a useful concept only when $R_0(\rho, \infty)$ does not change.
- The effect of initial conditions is only relevant as long as it provides time for policies to reduce $R_0(\rho, 0)$.
- Relatively rapid fluctuations in R_0 result into an effective value equal to the mean.
- Diffusion is necessary to initiate propagating infection waves.
- The wave velocity scales with the square root of diffusion coefficient and the inverse recovery time, and increases almost linearly with $R_0(\rho, 0)$.
- In 2-D heterogeneous systems, the wave solutions rapidly approach the asymptotic states corresponding to the ambient $R_0(\rho, 0)$.
- While here restricted to three species, the approach applies to additional species and demographics.