The Mathematical Theory of Epidemics

A Century-Long Saga

based on the Kermack-McKendrick (K-M or SIR) Model published in

Proceedings of the Royal Society, London, 1927

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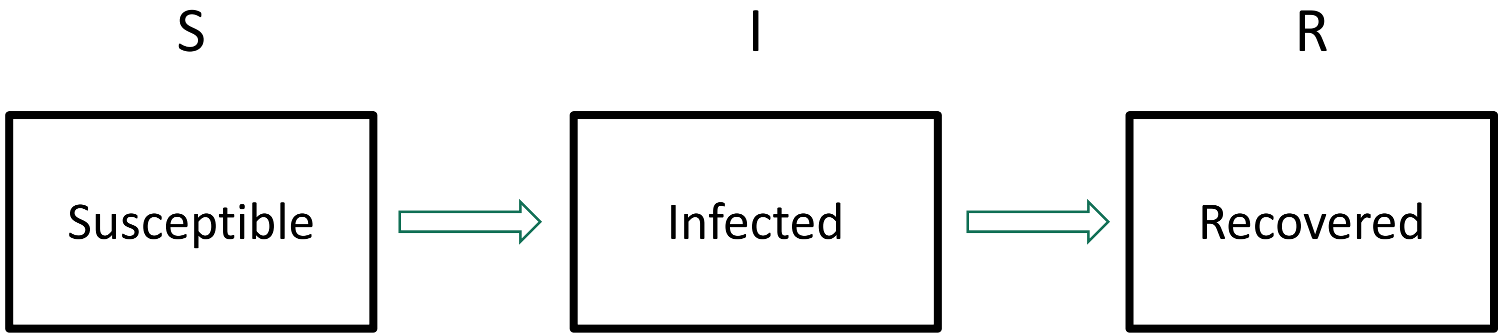
"As a matter of fact, all epidemiology, concerned as it is with the variation of disease from time to time or from place to place must be considered mathematically, if it is to be considered scientifically at all."

Sir Ronald Ross, M.D.

Second Scientist awarded the Nobel Prize in Medicine and Physiology (1902) for his discovery of the transmission of malaria by the mosquito. He was also a closet Mathematician and published papers in several areas of pure and applied mathematics.

Three Classes of Population

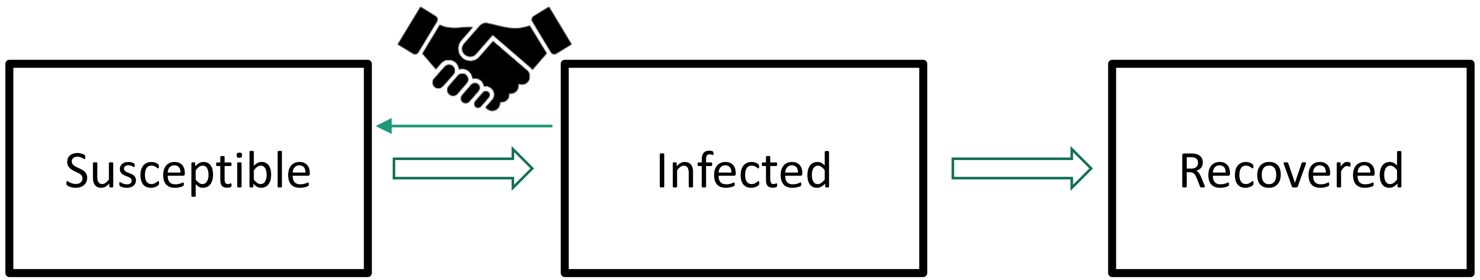


$$x(t) + y(t) + z(t) = N$$

Total Population

Initial Conditions

$x_0 = N - y_0$ $\quad$ y_0 $\quad$ $z_0 = 0$



Total Population

$x(t)$ + $y(t)$ + $z(t)$ = N

$$x'(t) = -\kappa \cdot x(t) \cdot y(t) \qquad (1)$$

$$y'(t) = \kappa \cdot x(t) \cdot y(t) - \lambda \cdot y(t) \qquad (2)$$

$$z'(t) = \lambda \cdot y(t) \qquad (3)$$

κ : contact rate

λ : recovery rate

$$\beta = \frac{\kappa}{\lambda}$$

$$\boldsymbol{R_o = \frac{N\kappa}{\lambda}}$$

Basic Reproductive Number

Dividing Eq.(1) by (3) and doing a little algebra ,

$$z'(t) = \lambda(N - x_o e^{-\beta z} - z)$$

This has no closed-form solution so replace exponential by first 3 terms of its Taylor Series expansion:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \ -\infty < x < \infty$$ [x is –βz in our case]

Result: $\frac{dz(t)}{dt} = \lambda y_o + \lambda(\ x_o\beta - 1)z - \left(\lambda x_o \frac{\beta^2}{2}\right) z^2$ $\quad first\ order\ quadratic\ differential\ equation$

Solution: $\boldsymbol{z(t)} = \frac{\boldsymbol{\lambda}}{\boldsymbol{\kappa^2 x_o}}[\boldsymbol{\lambda(\beta x_o - 1)} + \boldsymbol{\delta} \cdot \boldsymbol{tanh}\left(\frac{\boldsymbol{\delta}}{\boldsymbol{2}}\boldsymbol{t} - \boldsymbol{\phi}\right)]$

where $\delta = \lambda\sqrt{(\ x_o\ \beta - 1)^2 - 2x_o y_o \beta^2}$

and $\phi = tanh^{-1}[\frac{\lambda(\beta x_o - 1)}{\delta}]$

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\text{sech}(x) = \frac{2}{e^x + e^{-x}}$$

Simplifying by setting $y_o = 0$

$$\frac{z(t)}{N} = \frac{1}{R_o}\left(1 - \frac{1}{R_o}\right)\left[1 + tanh\left(\frac{(R_o - 1)}{2}\lambda t - \phi\right)\right]$$

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\frac{y(t)}{N} = \frac{\left(\frac{1}{\lambda}\right) z'(t)}{N} = \frac{1}{2}(1 - \frac{1}{R_o})^2 \left[sech^2\left(\frac{(R_o - 1)}{2}\lambda t - \phi\right)\right]$$

$$\text{sech}(x) = \frac{2}{e^x + e^{-x}}$$

$$\lim_{t \to \infty} \frac{z(t)}{N} = \frac{2}{R_o}(1 - \frac{1}{R_o})$$

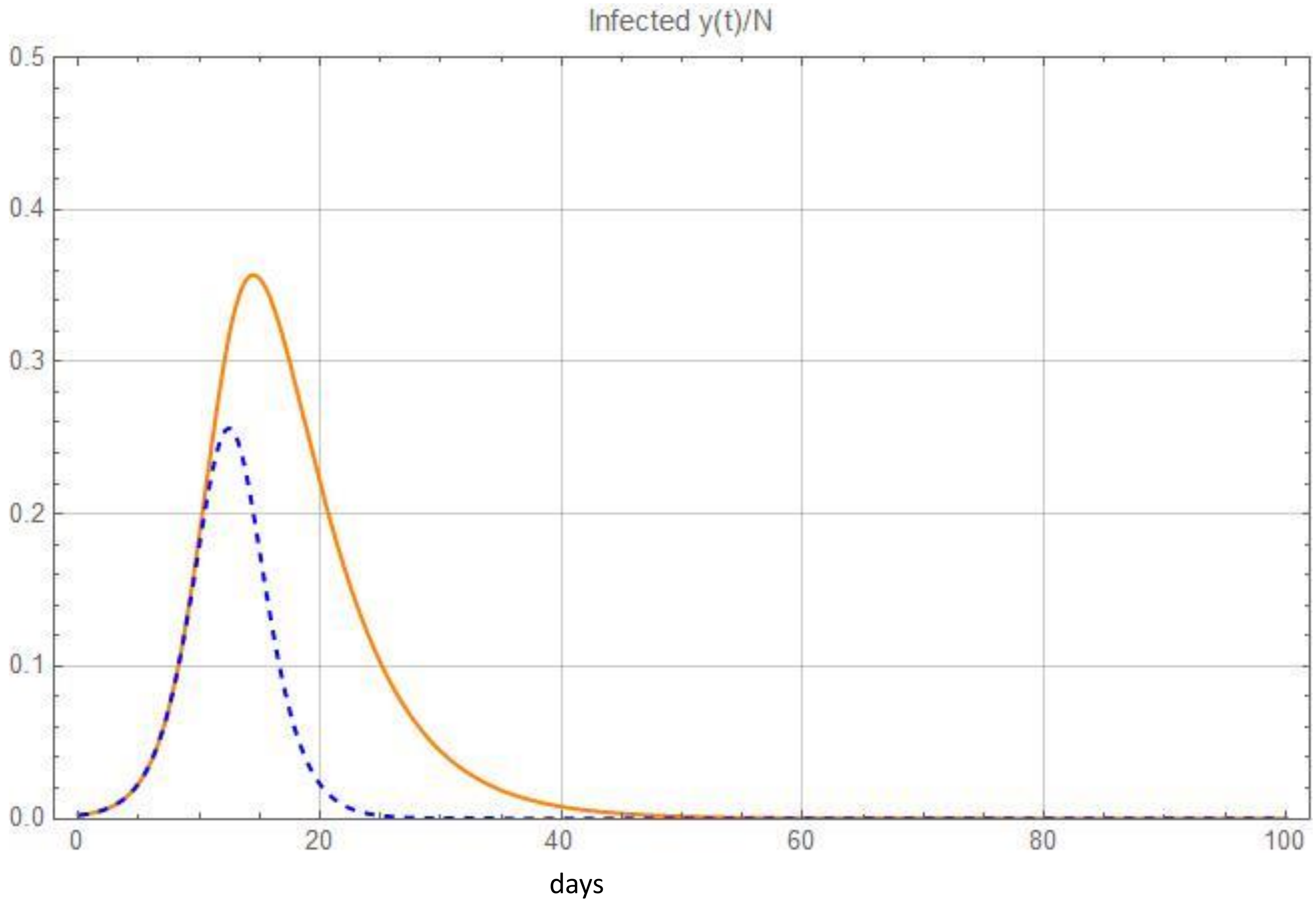
$$\max_t \frac{y(t)}{N} = \frac{1}{2}(1 - \frac{1}{R_o})^2$$

Only Graphs henceforth-

No More Equations.

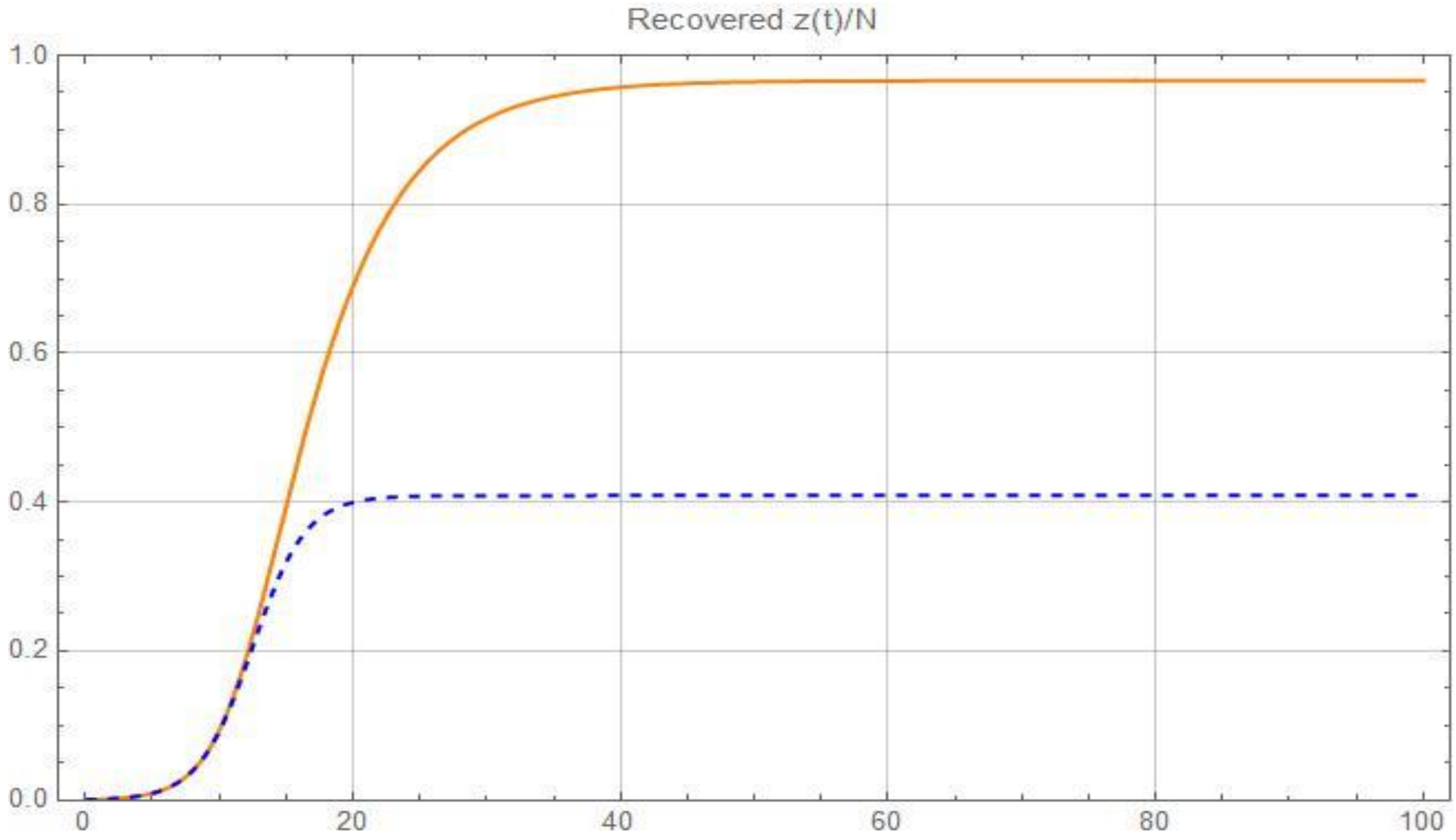
y(t)/N: Percentage of Infected (R_o =3.5)

Dashed: K-M   Solid: Numerical



z(t)/N: Percentage of Recovered (R_o=3.5)

Dashed: K-M Solid: Numerical

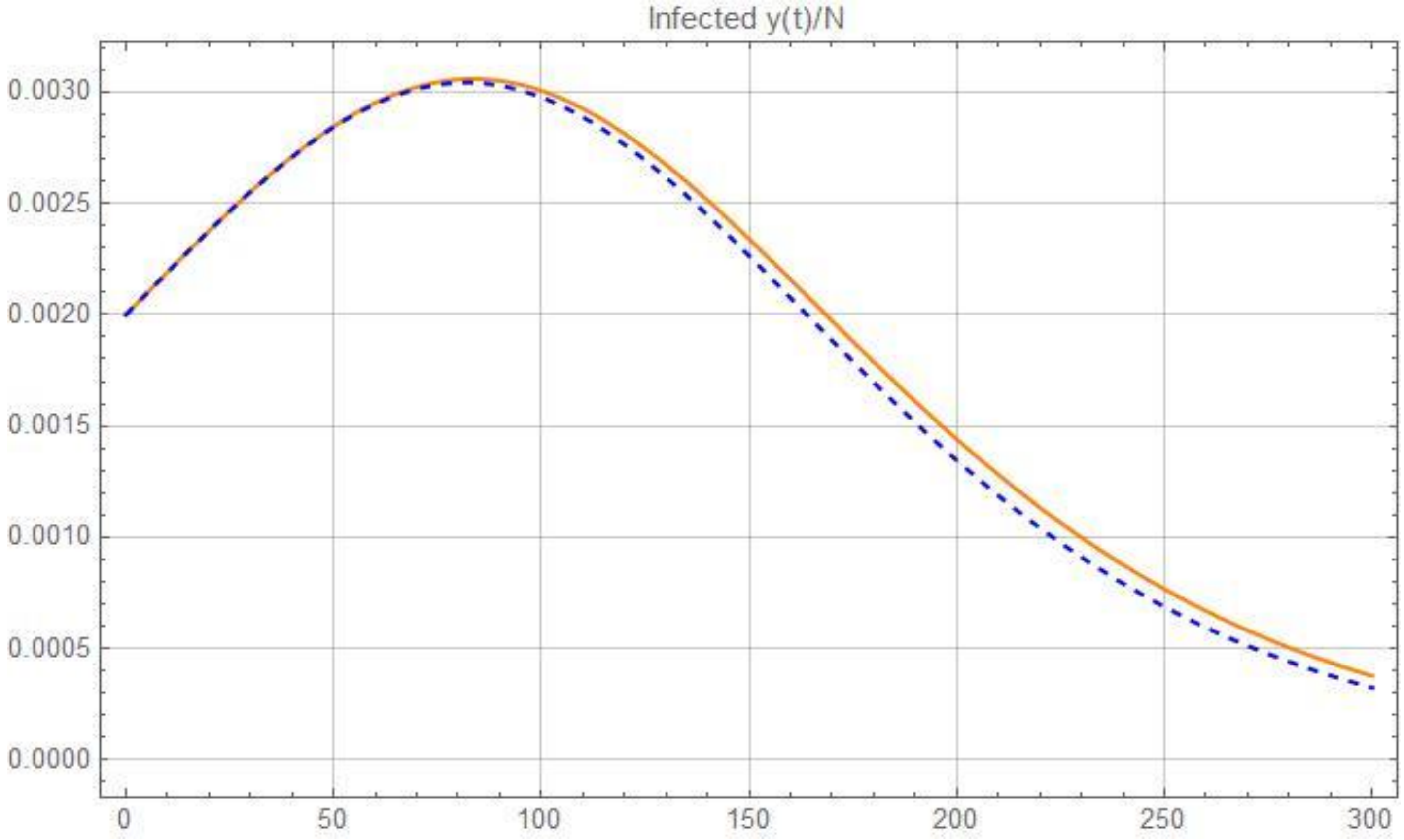


.965

.408

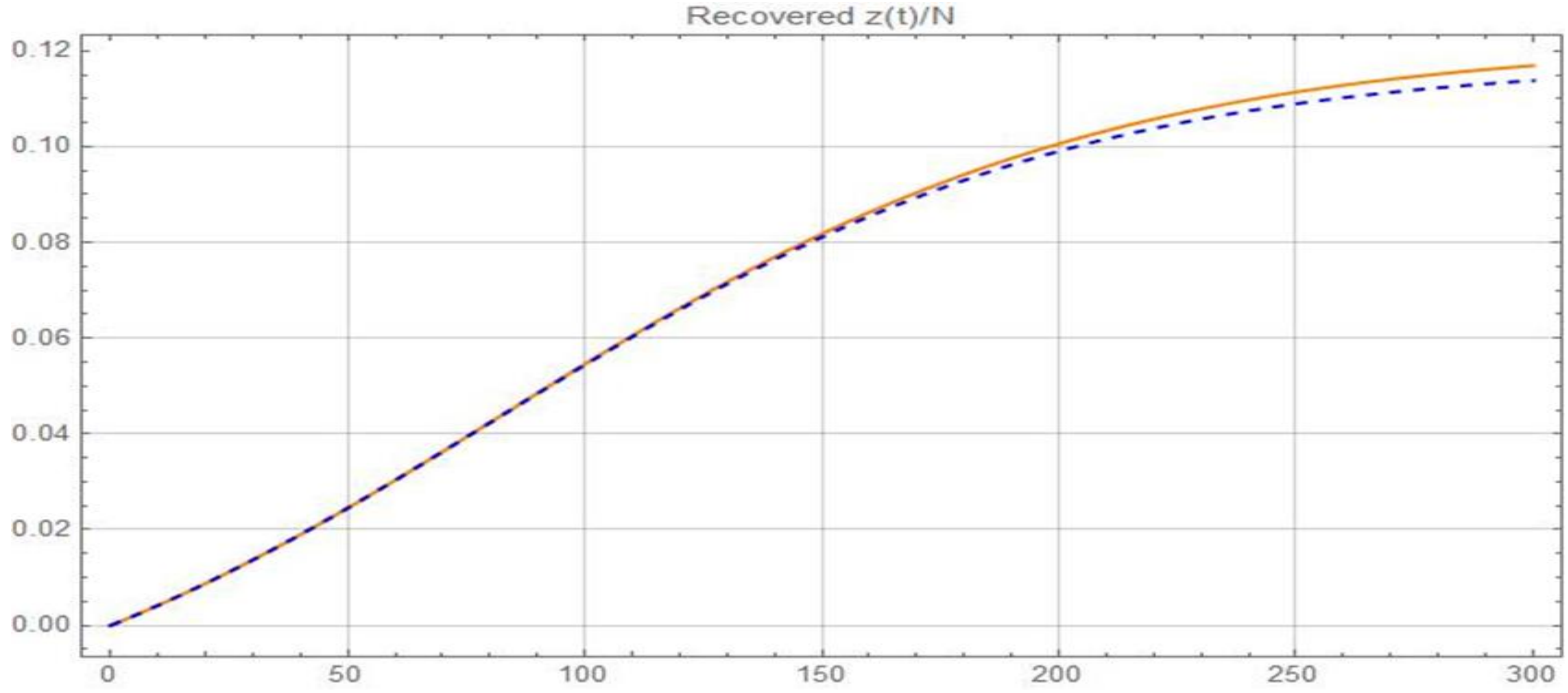
days

y(t)/N: Percentage of Infected (R_o=1.1)



days

z(t): Percentage of Recovered (R_o=1.1)



days

Comparison of Approximate (K-M) and True Analytic (+Numerical) Maxima and Limits

K-M Approximate

$$\max_t \frac{y(t)}{N} = \frac{1}{2}(1 - \frac{1}{R_o})^2$$

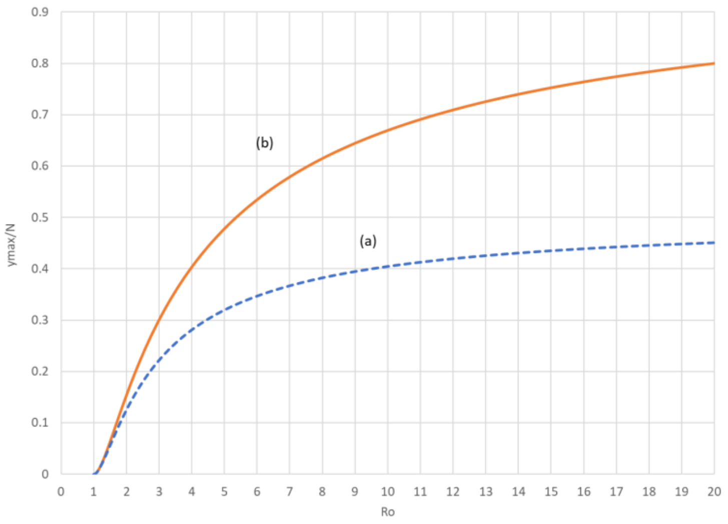
$$\lim_{t\to\infty} \frac{z(t)}{N} = \frac{2}{R_o}(1 - \frac{1}{R_o})$$

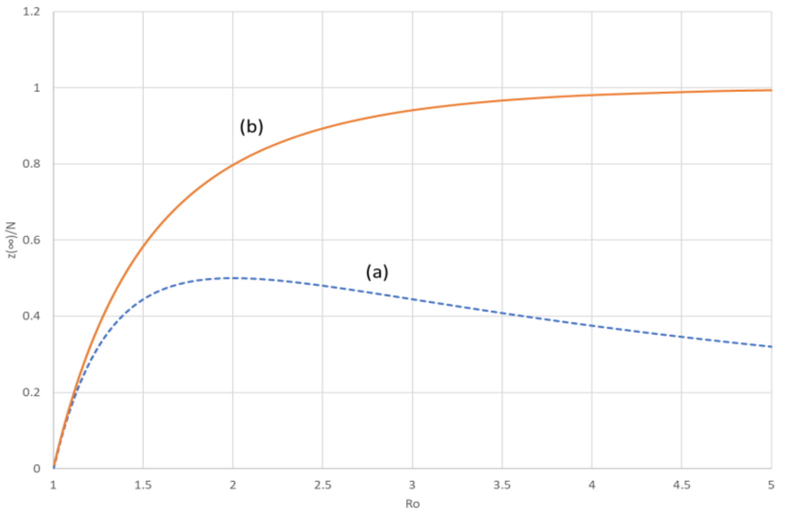
Accurate Analytic

$$\max_t \frac{y(t)}{N} = 1 - \frac{1}{R_o}(1 + \ln R_o)$$

$$e^{-R_o w} + w = 1\,; \quad w = \lim_{t\to\infty} z(t)/N.$$

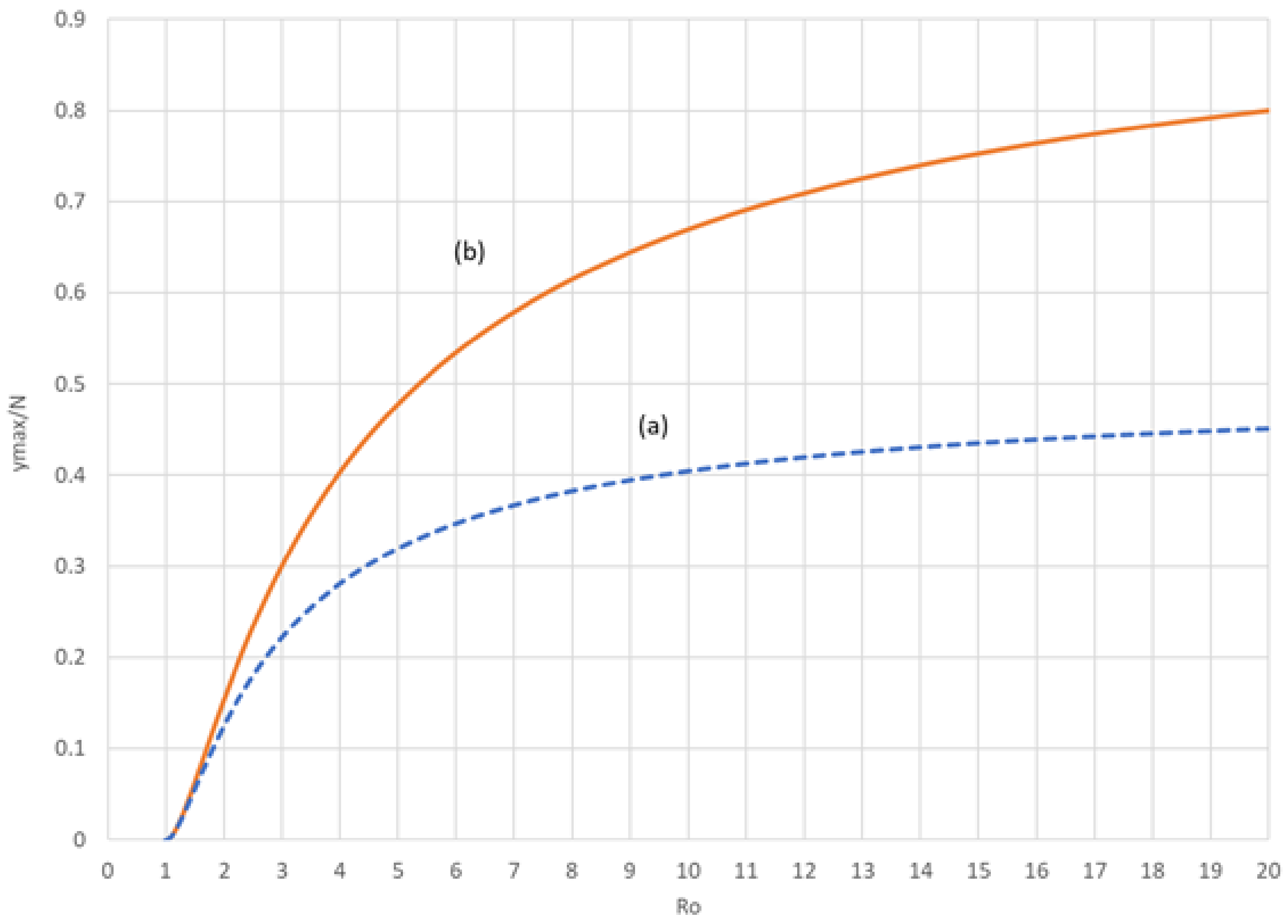
Max y(t)/N vs R_o



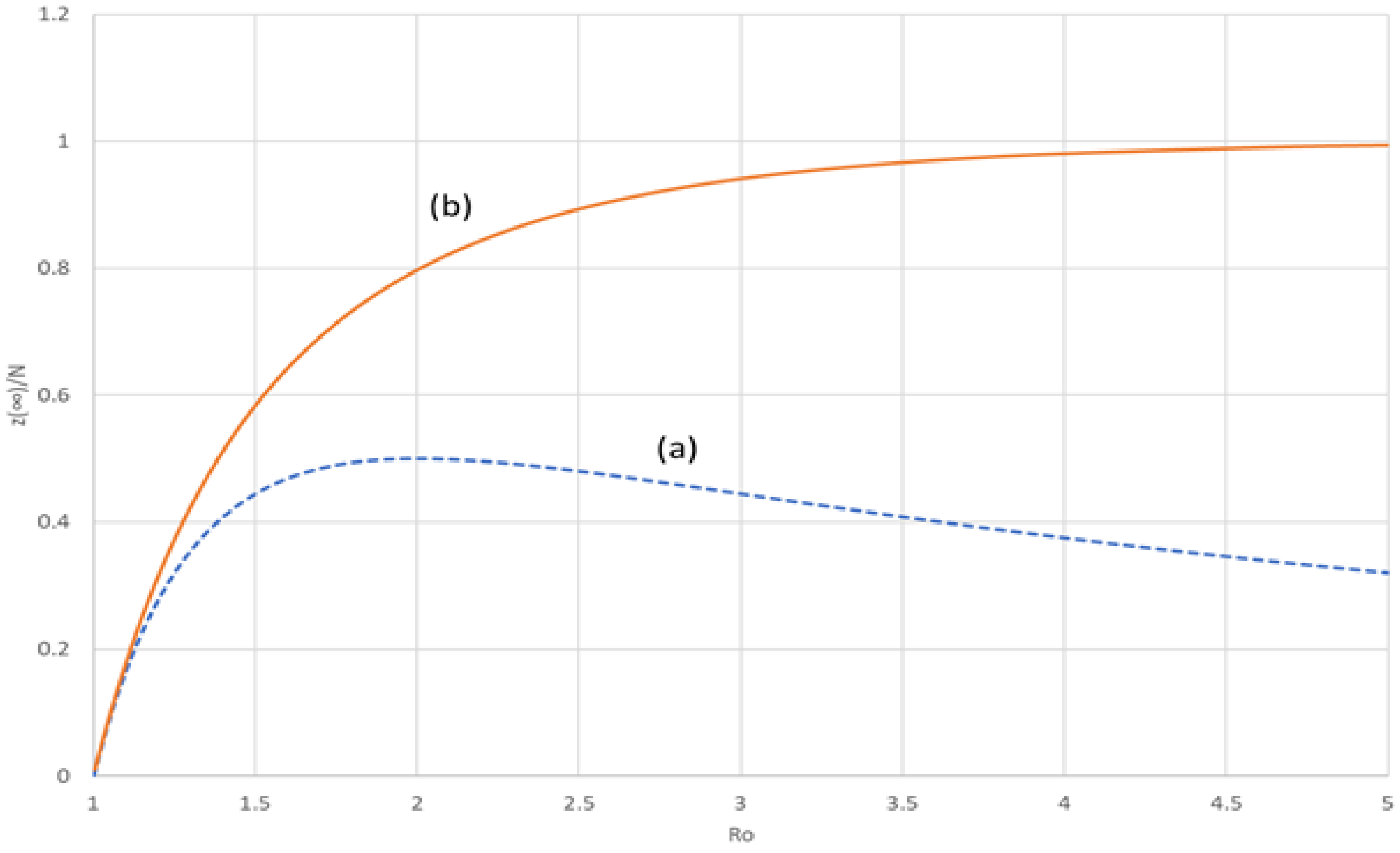


Lim z(t)/N vs R_o

Max y(t)/N: Infected Percentage vs R_o



Lim z(t)/N: Recovered Percentage vs R_o
t⟶∞



Numerical Solution: **Difference Equations**

$$x_{n+1} = x_n - \kappa x_n y_n$$

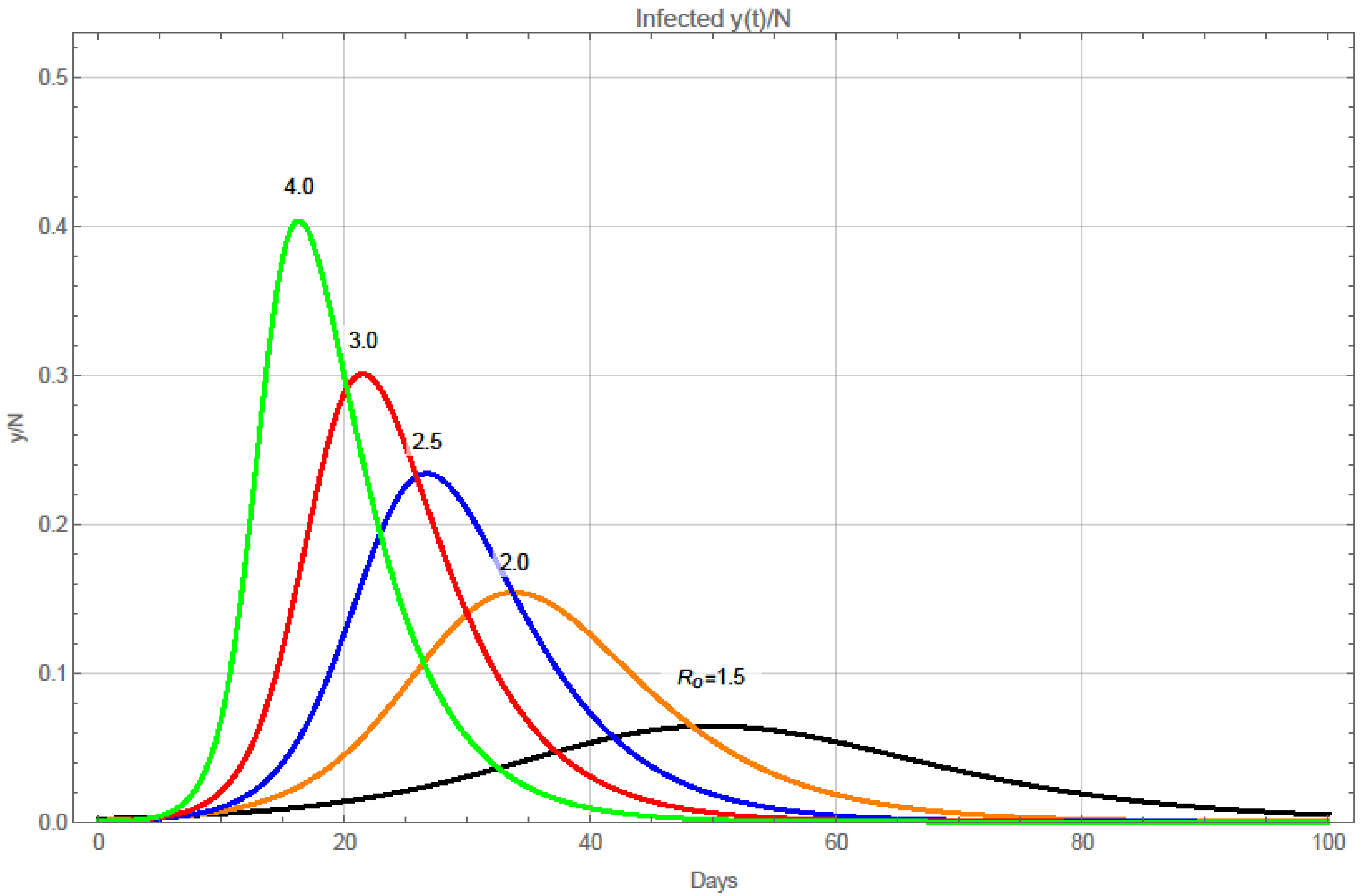
$$y_{n+1} = y_n + \kappa x_n y_n - \lambda y_n$$

$$z_{n+1} = z_n + \lambda y_n$$

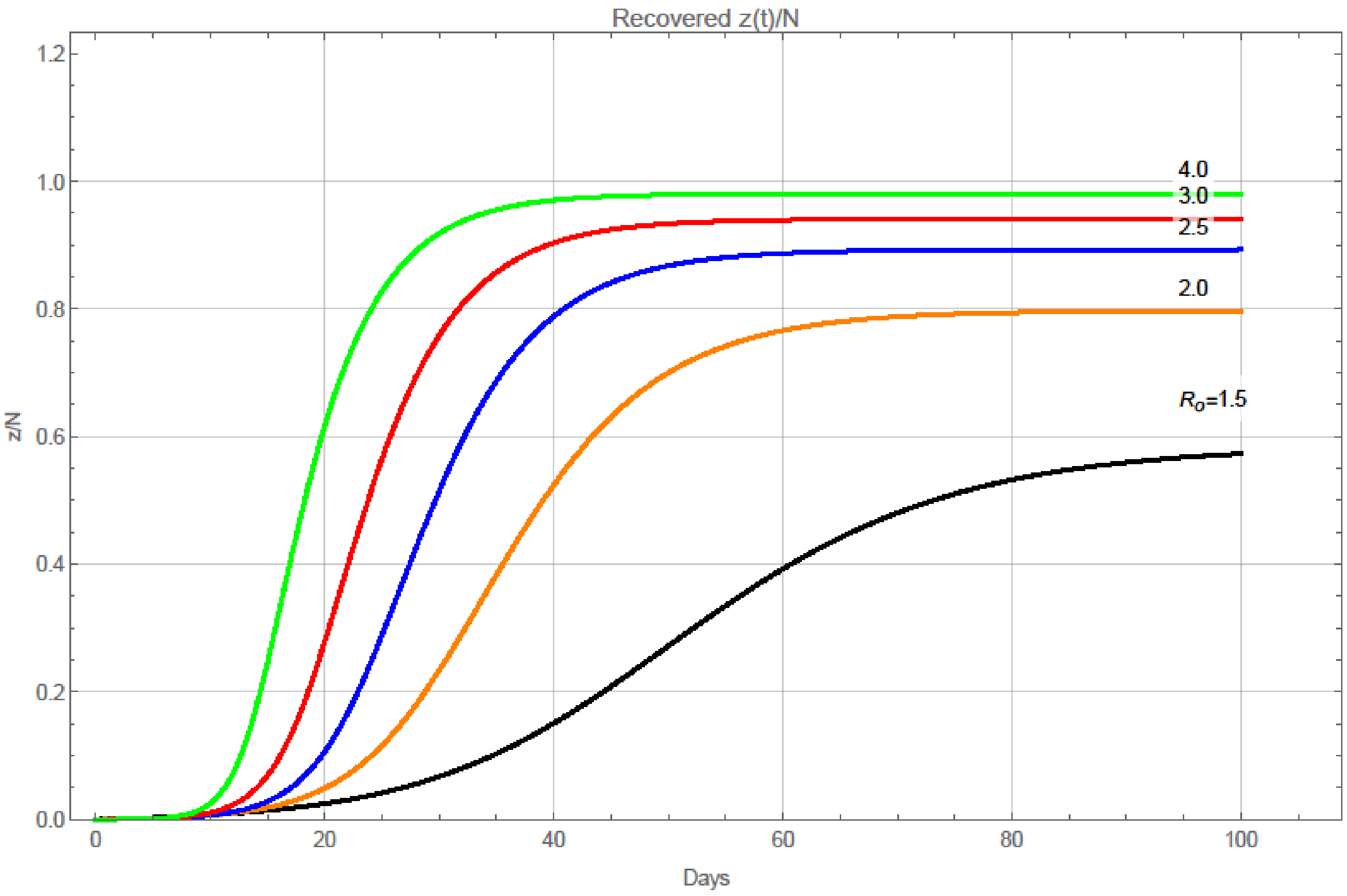
$$x_o = N - y_o;\ y_o = 5;\ z_o = 0$$

$$N = 5 * 10^4;\ \kappa * N = .7;\ \lambda = .2 \implies Ro = 3.5.$$

True Numerical **Infected Percentages** for R_o = 1.5 through 4.0



True Numerical **Recovered Percentages** for R_o = 1.5 through 4.0



Summary and Conclusions

- Kermack & McKendrick proposed a simple, logical model for the start and evolution of epidemics, which reveals significant features (infection peaks and ratios) and especially a metric for their intensity, R_o now used universally.

- Its value is in its generality and the very few parameters on which it depends (actually only three which can be rolled into one, $\mathbf{R_o}$).

- The principal weakness of the model is the assumption that the key parameters remain constant throughout. This is acceptable for λ, the recovery rate (inverse of recovery time), but not for κ, the infection rate, mostly because of variations in separation, regulation and isolation.

Closing Comment and Wisdom

Mathematics is all about creating models–
for Physics, Chemistry and Biology
But
A Model is not the Real Thing.

"Don't Eat the Menu", S.W. Golomb